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Universal R-matrices for non-standard $(1+1)$ quantum groups

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Abstract. A universal quasi-triangular R-matrix for the non-standard quantum $(1+1)$ Poincaré algebra $U_z iso(1,1)$ is deduced by imposing analyticity in the deformation parameter z. A family $U_w g_\mu$ of 'quantum graded contractions' of the algebra $U_z iso(1,1) \oplus U_{-z} iso(1,1)$ is obtained. Quantum analogues of the two-dimensional Euclidean, Poincaré and Galilei algebras enlarged with dilations are contained in $U_w g_\mu$ as Hopf subalgebras with two primitive translations. Universal R-matrices for these quantum Weyl (similitude) algebras and their associated quantum groups are constructed.

1. Introduction

Two types of quantum deformations for the $so(2,2)$ algebra and for its most relevant graded contractions have recently been studied in [1]. They are called *standard* and *non-standard* quantum algebras according to the fact that their corresponding coboundary Lie bi-algebras come from a classical r-matrix which is a skew solution either of the modified classical Yang–Baxter equation (YBE), or of the classical YBE respectively. In contradistinction with the standard case, the family of non-standard quantum algebras contains two-dimensional Euclidean and $(1+1)$ Poincaré and Galilei algebras enlarged with a dilation as quantum Hopf subalgebras: the so-called 'Weyl' or similitude subalgebras. These quantum subalgebras share the property of including the two translation generators as primitives. This fact could be relevant in relation to the problem of discretizing two-dimensional spaces in some symmetric way.

Let us also recall that a quasi-triangular Hopf algebra [2] is a pair $(\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a Hopf algebra and $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ is invertible and verifies that

$$\sigma \circ \Delta X = \mathcal{R}(\Delta X)\mathcal{R}^{-1} \qquad \forall\, X \in \mathcal{A} \tag{1.1}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{1.2}$$

where, if $\mathcal{R} = \sum_i a_i \otimes b_i$, we denote $\mathcal{R}_{12} \equiv \sum_i a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13} \equiv \sum_i a_i \otimes 1 \otimes b_i$, $\mathcal{R}_{23} \equiv \sum_i 1 \otimes a_i \otimes b_i$ and σ is the flip operator $\sigma(x \otimes y) = (y \otimes x)$. If $\mathcal{A}$ is a quasi-triangular Hopf algebra then $\mathcal{R}$ is called a universal R-matrix and satisfies the quantum YBE

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1.3}$$

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In this paper the construction of universal R-matrices and quantum groups for the above mentioned family of non-standard quantum algebras is discussed. In particular, we obtain the universal R-matrices for the Weyl subalgebras.

A straightforward approach to this problem would start from a universal R-matrix for the non-standard Hopf algebra $U_z sl(2,\mathbb{R}) \simeq U_z so(2,1)$, since the prescription $U_z so(2,2) \simeq U_z so(2,1) \oplus U_{-z} so(2,1)$ applied to R-matrices would lead to a universal R-matrix for $U_z so(2,2)$. By introducing 'quantum graded contractions' a set of R-matrices for all the family of non-standard algebras would then be obtained. A similar procedure was developed in [3,4] for obtaining universal R-matrices for some standard quantum algebras. Unfortunately, to our knowledge, no universal R-matrix for the non-standard $U_z sl(2,\mathbb{R})$ has appeared in any literature; the universal R-matrix given in [5] neither verifies (1.1) nor satisfies (1.3).

Therefore, we have to focus on the problem from a different point of view. Following the method developed in [6] for the standard $(1+1)$ groups and in [7] for the Heisenberg group, we impose analyticity in the deformation parameter z and relation (1.1) in order to obtain an R-matrix for the non-standard quantum $(1+1)$ Poincaré algebra $U_z iso(1,1)$. The R-matrix so obtained coincides in turn with a universal R-matrix for the positive Borel subalgebra of the non-standard $sl(2,\mathbb{R})$ given in [8]; this fact proves its universality. This explicit construction together with a brief overview of both the quantum Poincaré algebra and the group studied in [9] is presented in section 2. The fact that

$$U_z t_4(so(1,1) \oplus so(1,1)) \simeq U_z iso(1,1) \oplus U_{-z} iso(1,1) \tag{1.4}$$

can also be used at the group and R-matrix levels and leads to the whole quantum structure for this group as shown in section 3; Lie algebras with structure $t_n(so(p,q) \oplus so(r,s))$ are described in [10]. A 'quantum graded contraction' is introduced providing quantum structures for two more non-standard quantum real algebras: $U_z iiso(1,1)$ and $U_z t_4(so(2) \oplus so(1,1))$ (the latter is isomorphic to the $(2+1)$ expanding Newton–Hooke algebra). Each of these quantum algebras contains a Hopf Weyl subalgebra. Section 4 is devoted to obtaining the universal R-matrices and quantum groups corresponding to these Weyl subalgebras.

2. Universal R-matrix for the Poincaré group

The $(1+1)$ Poincaré algebra $iso(1,1)$ is generated by one boost generator K and the translation generators along the light-cone $P_\pm$. The Lie brackets are

$$[K, P_\pm] = \pm 2P_\pm \qquad [P_+, P_-] = 0. \tag{2.1}$$

A non-standard coboundary bi-algebra of $iso(1,1)$ is generated by the classical r-matrix $r = zK \wedge P_+$ which verifies the classical YBE.

The quantum deformations for the universal enveloping algebra $Uiso(1,1)$ and for the algebra of smooth functions on the group $Fun(ISO(1,1))$, denoted respectively by $U_z iso(1,1)$ and $Fun_z(ISO(1,1))$, are given by the following propositions (see [9] for a more detailed exposition and proofs, and also [11]).

Proposition 1. The Hopf structure of $U_z iso(1,1)$ is given by the coproduct, co-unit and antipode

$$\begin{aligned} \Delta P_+ &= 1 \otimes P_+ + P_+ \otimes 1 \\ \Delta P_- &= e^{-zP_+} \otimes P_- + P_- \otimes e^{zP_+} \\ \Delta K &= e^{-zP_+} \otimes K + K \otimes e^{zP_+} \end{aligned} \tag{2.2}$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{zP_+} X e^{-zP_+} \qquad \text{for } X \in \{K, P_\pm\} \tag{2.3}$$

and the commutation relations

$$[K, P_+] = 2\frac{\sinh zP_+}{z} \qquad [K, P_-] = -2P_- \cosh zP_+ \qquad [P_+, P_-] = 0. \tag{2.4}$$

Proposition 2. The Hopf algebra $Fun_z(ISO(1,1))$ has multiplication given by

$$[\hat{\chi}, \hat{a}_+] = z(e^{2\hat{\chi}} - 1) \qquad [\hat{\chi}, \hat{a}_-] = 0 \qquad [\hat{a}_+, \hat{a}_-] = -2z\hat{a}_- \tag{2.5}$$

coproduct

$$\Delta(\hat{\chi}) = \hat{\chi} \otimes 1 + 1 \otimes \hat{\chi} \qquad \Delta(\hat{a}_\pm) = \hat{a}_\pm \otimes 1 + e^{\pm 2\hat{\chi}} \otimes \hat{a}_\pm \tag{2.6}$$

co-unit and antipode

$$\epsilon(X) = 0 \qquad X \in \{\hat{a}_+, \hat{a}_-, \hat{\chi}\} \tag{2.7}$$

$$\gamma(\hat{\chi}) = -\hat{\chi} \qquad \gamma(\hat{a}_\pm) = -e^{\mp 2\hat{\chi}}\hat{a}_\pm. \tag{2.8}$$

The quantum coordinates $\hat{\chi}$, $\hat{a}_-$ and $\hat{a}_+$ of $Fun_z(ISO(1,1))$ are, respectively, the dual basis of the $U_z iso(1,1)$ generators $H = e^{zP_+}K$, $A_- = e^{-zP_+}P_-$ and $A_+ = P_+$.

We now proceed to deduce a universal R-matrix for $U_z iso(1,1)$. We assume that the R-matrix is analytical in the quantum parameter z $(= \log q)$ and that $R = 1 \otimes 1 + zK \wedge P_+ + O(z^2)$; hence, we consider an R-matrix as a formal power series in z with coefficients in $Uiso(1,1) \otimes Uiso(1,1)$. We start from the ansatz

$$R = \exp\{zf(K, P_+, z)g(P_+, P_-, z)\}. \tag{2.9}$$

Firstly, we impose R to verify relation (1.1). Starting with the primitive generator P_+, it is implied that

$$[R, \Delta P_+] = 0. \tag{2.10}$$

This requirement is fulfilled if

$$[f(K, P_+, z)g(P_+, P_-, z), \Delta P_+] = [f(K, P_+, z), \Delta P_+]g(P_+, P_-, z) = 0. \tag{2.11}$$

Therefore, by taking into account commutation rules (2.4) a solution for $f(K, P_+, z)$ is

$$f(K, P_+, z) = K \wedge \sinh zP_+. \tag{2.12}$$

We should now apply condition (1.1) for the two remaining generators P_- and K. Omitting the arguments of the functions f and g we have

$$R\Delta X R^{-1} = \exp(zfg)\Delta X \exp(-zfg) = \Delta X + z[fg, \Delta X] + \frac{z^2}{2!}[fg, [fg, \Delta X]] + \cdots$$
$$+ \frac{z^n}{n!}[fg, [fg \cdots [fg, \Delta X]^{n)} \cdots]] + \cdots. \tag{2.13}$$

Since P_- commutes with P_+, relation (2.13) with $X \equiv P_-$ becomes

$$\exp(zfg)\Delta P_- \exp(-zfg) = \Delta P_- + z[f, \Delta P_-]g + \frac{z^2}{2!}[f, [f, \Delta P_-]]g^2 + \cdots$$
$$+ \frac{z^n}{n!}[f, [f \cdots [f, \Delta P_-]^{n)} \cdots]]g^n + \cdots. \tag{2.14}$$

We need to obtain the brackets $[f, \Delta P_-]$, $[f, [f, \Delta P_-]], \ldots$ in (2.14). The first and the second brackets are

$$[f, \Delta P_-] = A \qquad [f, [f, \Delta P_-]] = B 2 \sinh z\Delta P_+ \tag{2.15}$$

where

$$A \equiv 2\exp(-z\Delta P_+)\sinh zP_+ \otimes P_- - 2\exp(z\Delta P_+)P_- \otimes \sinh zP_+ \tag{2.16}$$

$$B \equiv 2\exp(-z\Delta P_+)\sinh zP_+ \otimes P_- + 2\exp(z\Delta P_+)P_- \otimes \sinh zP_+. \tag{2.17}$$

Due to expression (2.12) f commutes with any arbitrary function of P_+, so we obtain

$$[f, \sinh z\Delta P_+] = 0 \qquad [f, B] = A 2 \sinh z\Delta P_+. \tag{2.18}$$

By a recurrence method we obtain the $2n$ and $2n+1$ iterates

$$[f, [f \cdots [f, \Delta P_-]^{2n)} \cdots]] = B(2\sinh z\Delta P_+)^{2n-1} \tag{2.19}$$

$$[f, [f \cdots [f, \Delta P_-]^{2n+1)} \cdots]] = A(2\sinh z\Delta P_+)^{2n} \tag{2.20}$$

so that expression (2.14) can be written as

$$\begin{aligned}\exp(zfg)\Delta P_- \exp(-zfg) = \Delta P_- + A\sum_{l=0}^{\infty}\frac{z^{2l+1}}{(2l+1)!}(2\sinh z\Delta P_+)^{2l}g^{2l+1} \\ +B\sum_{l=1}^{\infty}\frac{z^{2l}}{(2l)!}(2\sinh z\Delta P_+)^{2l-1}g^{2l} \\ = \Delta P_- + \frac{A}{2\sinh z\Delta P_+}\sinh(2z\sinh(z\Delta P_+)g) \\ +\frac{B}{2\sinh z\Delta P_+}\left[\cosh(2z\sinh(z\Delta P_+)g) - 1\otimes 1\right].\end{aligned} \tag{2.21}$$

By introducing

$$C = \frac{1}{2\sinh z\Delta P_+}\sinh(2z\sinh(z\Delta P_+)g) \tag{2.22}$$

$$D = \frac{1}{2\sinh z\Delta P_+}\left[\cosh(2z\sinh(z\Delta P_+)g) - 1\otimes 1\right] \tag{2.23}$$

the right-hand side of expression (2.21) can be written as $\Delta P_- + AC + BD$, which must be equal to

$$\sigma\circ\Delta P_- = e^{zP_+}\otimes P_- + P_-\otimes e^{-zP_+}. \tag{2.24}$$

Thus, if we impose that (2.21) coincides with (2.24) we obtain the following system of equations for the function g:

$$(1\otimes P_-)(e^{-zP_+}\otimes 1 - e^{zP_+}\otimes 1 + 2e^{-zP_+}\sinh zP_+\otimes e^{-zP_+}(D+C)) = 0 \tag{2.25}$$

$$(P_-\otimes 1)(1\otimes e^{zP_+} - 1\otimes e^{-zP_+} + 2e^{zP_+}\sinh zP_+\otimes e^{zP_+}(D-C)) = 0. \tag{2.26}$$

Both equations can be summarized by the expression

$$\exp(\pm 2z\sinh(z\Delta P_+)g) = \exp(\pm 2z\Delta P_+) \tag{2.27}$$

leading to the same result for the function g:

$$g = \frac{\Delta P_+}{\sinh z\Delta P_+}. \tag{2.28}$$

Finally, an explicit check shows that the R-matrix

$$R = \exp\left\{K\wedge\sinh zP_+\frac{z\Delta P_+}{\sinh z\Delta P_+}\right\} \tag{2.29}$$

also verifies property (1.1) for the last generator K. Note that the functions f (2.12) and g (2.28) commute.

Result (2.29) is in fact similar to a universal R-matrix given in [8] for a Hopf algebra $\{v, h\}$ which is isomorphic to the Hopf subalgebra $\{K, P_+\}$ of $U_z iso(1,1)$. Therefore, since the R-matrix (2.29) does not depend on P_-, the universality holds and (2.29) satisfies the quantum YBE (1.3). It is worth remarking that expression (2.29) has also been obtained in [12] following a different procedure.

3. Construction of $U_z iso(1,1) \oplus U_{-z} iso(1,1)$

Let us consider two copies of $iso(1,1)$ with generators $\{K^l, P_\pm^l\}$ $(l = 1, 2)$. The set of generators defined by

$$J_3 = K^1 + K^2 \qquad J_\pm = P_\pm^1 + P_\pm^2 \qquad N_3 = K^1 - K^2 \qquad N_\pm = P_\pm^1 - P_\pm^2 \tag{3.1}$$

closes the algebra $t_4(so(1,1) \oplus so(1,1)) \simeq iso(1,1) \oplus iso(1,1)$. The formal transformation (equivalent to a graded contraction [1]) defined by

$$(J, P_1, P_2, C_1, C_2, D) := (\sqrt{\mu}\, N_3/2, J_+, \sqrt{\mu}\, N_+, -J_-, \sqrt{\mu}\, N_-, J_3/2) \tag{3.2}$$

gives rise to the non-vanishing commutation relations

$$\begin{aligned} &[J, P_1] = P_2 \qquad [J, P_2] = \mu P_1 \qquad [D, P_i] = P_i \\ &[J, C_1] = C_2 \qquad [J, C_2] = \mu C_1 \qquad [D, C_i] = -C_i. \end{aligned} \tag{3.3}$$

For μ equal to $+1$, 0 and -1 we obtain the commutators of $t_4(so(1,1) \oplus so(1,1))$, $iiso(1,1)$ and $t_4(so(2) \oplus so(1,1))$ respectively. We denote these three algebras by g_μ.

We will now show how the results presented in the previous section for the quantum non-standard (1 + 1) Poincaré algebra provide a quantum structure for the algebras g_μ and for the groups G_μ as well as their universal R-matrices.

The invariance of $U_z iso(1,1)$ under the transformation $z \to -z$ allows us to write $U_z t_4(so(1,1) \oplus so(1,1)) = U_z iso(1,1) \oplus U_{-z} iso(1,1)$. The contraction (3.3) is implemented in the quantum case by considering the following definition of the contracted generators and deformation parameter:

$$(J, P_1, P_2, C_1, C_2, D; w) := (\sqrt{\mu}\, N_3/2, J_+, \sqrt{\mu}\, N_+, -J_-, \sqrt{\mu}\, N_-, J_3/2; z/\sqrt{\mu}) \tag{3.4}$$

where w is the new (contracted) quantum parameter. In this way we obtain the Hopf structure of $U_w g_\mu$. We omit the explicit expressions so obtained since they are exactly the quantum algebras $U_w^{(n)} g_{(\mu_1, 0, +)}$ (with $\mu \equiv \mu_1$) given in [1].

3.1. Poisson–Hopf structure of $Fun(G_\mu)$

Before obtaining the quantum groups associated to $U_w g_\mu$ we first study the algebra $Fun(G_\mu)$ of smooth functions on the group G_μ.

A matrix realization of g_μ in terms of 4×4 real matrices is

$$J = \begin{pmatrix} 0 & -\mu & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Hence, a real 4×4 representation of the element $g = e^{c_1 C_1} e^{c_2 C_2} e^{p_1 P_1} e^{p_2 P_2} e^{dD} e^{\theta J} \in G_\mu$ is given by

$$g = \begin{pmatrix} \mathrm{C}_{-\mu}(\theta) & -\mu \mathrm{S}_{-\mu}(\theta) & 0 & 0 \\ -\mathrm{S}_{-\mu}(\theta) & \mathrm{C}_{-\mu}(\theta) & 0 & 0 \\ t_{31} & t_{32} & \cosh d & \sinh d \\ t_{41} & t_{42} & \sinh d & \cosh d \end{pmatrix} \tag{3.6}$$

with

$$\begin{aligned} t_{31} &= (p_2 - c_2)\mathrm{C}_{-\mu}(\theta) - (p_1 - c_1)\mathrm{S}_{-\mu}(\theta) \\ t_{32} &= (p_1 - c_1)\mathrm{C}_{-\mu}(\theta) - \mu(p_2 - c_2)\mathrm{S}_{-\mu}(\theta) \\ t_{41} &= (p_2 + c_2)\mathrm{C}_{-\mu}(\theta) - (p_1 + c_1)\mathrm{S}_{-\mu}(\theta) \\ t_{42} &= (p_1 + c_1)\mathrm{C}_{-\mu}(\theta) - \mu(p_2 + c_2)\mathrm{S}_{-\mu}(\theta). \end{aligned} \tag{3.7}$$

The generalized sine and cosine functions are defined by

$$\mathrm{C}_{-\mu}(\theta) = \frac{\mathrm{e}^{\sqrt{\mu}\theta} + \mathrm{e}^{-\sqrt{\mu}\theta}}{2} \qquad \mathrm{S}_{-\mu}(\theta) = \frac{\mathrm{e}^{\sqrt{\mu}\theta} - \mathrm{e}^{-\sqrt{\mu}\theta}}{2\sqrt{\mu}}. \tag{3.8}$$

Note that for μ equal to $+1$ and -1 we recover the hyperbolic and elliptic trigonometric functions. The case $\mu = 0$ corresponds to a contraction of the group representation (3.6): $\mathrm{C}_0(\theta) = 1$ and $\mathrm{S}_0(\theta) = \theta$.

Proposition 3. The fundamental Poisson brackets

$$\begin{aligned} &\{d, p_1\} = w\mu \mathrm{e}^d \mathrm{S}_{-\mu}(\theta) && \{p_1, c_1\} = w\mu c_2 \\ &\{d, p_2\} = w(\mathrm{e}^d \mathrm{C}_{-\mu}(\theta) - 1) && \{p_1, c_2\} = w c_1 \\ &\{\theta, p_1\} = w(\mathrm{e}^d \mathrm{C}_{-\mu}(\theta) - 1) && \{p_2, c_1\} = -w c_1 \\ &\{\theta, p_2\} = w \mathrm{e}^d \mathrm{S}_{-\mu}(\theta) && \{p_2, c_2\} = -w c_2 \end{aligned} \tag{3.9}$$

endow $Fun(G_\mu)$ with a Poisson–Hopf algebra structure.

The Poisson brackets (3.9) are obtained from the Sklyanin bracket induced from a classical r-matrix

$$\{\Psi, \Phi\} = r^{\alpha\beta}(X_\alpha^L \Psi X_\beta^L \Phi - X_\alpha^R \Psi X_\beta^R \Phi) \qquad \Psi, \Phi \in Fun(G_\mu). \tag{3.10}$$

In our case the r-matrix which satisfies the classical YBE is given by

$$r = w(J \wedge P_1 + D \wedge P_2) \tag{3.11}$$

while left- and right-invariant vector fields are deduced from (3.6):

$$\begin{aligned} &X_J^L = \partial_\theta \qquad X_D^L = \partial_d \\ &X_{P_1}^L = \mathrm{e}^d \mathrm{C}_{-\mu}(\theta)\partial_{p_1} + \mathrm{e}^d \mathrm{S}_{-\mu}(\theta)\partial_{p_2} \\ &X_{P_2}^L = \mathrm{e}^d \mathrm{C}_{-\mu}(\theta)\partial_{p_2} + \mu \mathrm{e}^d \mathrm{S}_{-\mu}(\theta)\partial_{p_1} \\ &X_{C_1}^L = \mathrm{e}^{-d} \mathrm{C}_{-\mu}(\theta)\partial_{c_1} + \mathrm{e}^{-d} \mathrm{S}_{-\mu}(\theta)\partial_{c_2} \\ &X_{C_2}^L = \mathrm{e}^{-d} \mathrm{C}_{-\mu}(\theta)\partial_{c_2} + \mu \mathrm{e}^{-d} \mathrm{S}_{-\mu}(\theta)\partial_{c_1} \end{aligned} \tag{3.12}$$

$$\begin{aligned} &X_J^R = \partial_\theta + \mu p_2 \partial_{p_1} + p_1 \partial_{p_2} + \mu c_2 \partial_{c_1} + c_1 \partial_{c_2} \\ &X_D^R = \partial_d + p_1 \partial_{p_1} + p_2 \partial_{p_2} - c_1 \partial_{c_1} - c_2 \partial_{c_2} \\ &X_{P_1}^R = \partial_{p_1} \qquad X_{P_2}^R = \partial_{p_2} \qquad X_{C_1}^R = \partial_{c_1} \qquad X_{C_2}^R = \partial_{c_2}. \end{aligned} \tag{3.13}$$

3.2. *Hopf structure of* $Fun_w(G_\mu)$

We now proceed to quantize the Poisson–Hopf algebra $Fun(G_\mu)$. First we consider two sets of quantum coordinates $\{\hat{\chi}^l, \hat{a}_+^l, \hat{a}_-^l\}$ $(l = 1, 2)$ of $Fun_z(ISO(1,1))$ for $l = 1$ and of $Fun_{-z}(ISO(1,1))$ for $l = 2$; then we construct the Hopf algebra $Fun_z(ISO(1,1)) \oplus$

$Fun_{-z}(ISO(1,1))$ with the results of proposition 2 and by using the new coordinates defined by

$$\hat{a} = \tfrac{1}{2}(\hat{\chi}^1 + \hat{\chi}^2) \qquad \hat{a}_\pm = \tfrac{1}{2}(\hat{a}^1_\pm + \hat{a}^2_\pm) \qquad \hat{b} = \tfrac{1}{2}(\hat{\chi}^1 - \hat{\chi}^2) \qquad \hat{b}_\pm = \tfrac{1}{2}(\hat{a}^1_\pm - \hat{a}^2_\pm). \tag{3.14}$$

Next we apply the quantum contraction induced at the group level from (3.4):

$$(\hat{\theta}, \hat{p}_1, \hat{p}_2, \hat{c}_1, \hat{c}_2, \hat{d}; w) := (2\hat{b}/\sqrt{\mu}, \hat{a}_+, \hat{b}_+/\sqrt{\mu}, -\hat{a}_-, \hat{b}_-/\sqrt{\mu}, 2\hat{a}; z/\sqrt{\mu}) \tag{3.15}$$

obtaining in this way the quantization of $Fun(G_\mu)$. The final result is summarized as follows.

Proposition 4. The Hopf algebra $Fun_w(G_\mu)$ is given by the non-vanishing commutators

$$\begin{aligned}
&[\hat{d}, \hat{p}_1] = w\mu e^{\hat{d}} S_{-\mu}(\hat{\theta}) && [\hat{p}_1, \hat{c}_1] = w\mu\hat{c}_2 \\
&[\hat{d}, \hat{p}_2] = w(e^{\hat{d}} C_{-\mu}(\hat{\theta}) - 1) && [\hat{p}_1, \hat{c}_2] = w\hat{c}_1 \\
&[\hat{\theta}, \hat{p}_1] = w(e^{\hat{d}} C_{-\mu}(\hat{\theta}) - 1) && [\hat{p}_2, \hat{c}_1] = -w\hat{c}_1 \\
&[\hat{\theta}, \hat{p}_2] = w e^{\hat{d}} S_{-\mu}(\hat{\theta}) && [\hat{p}_2, \hat{c}_2] = -w\hat{c}_2
\end{aligned} \tag{3.16}$$

coproduct, co-unit and antipode

$$\begin{aligned}
&\Delta(\hat{\theta}) = \hat{\theta} \otimes 1 + 1 \otimes \hat{\theta} \qquad \Delta(\hat{d}) = \hat{d} \otimes 1 + 1 \otimes \hat{d} \\
&\Delta(\hat{p}_1) = \hat{p}_1 \otimes 1 + e^{\hat{d}} C_{-\mu}(\hat{\theta}) \otimes \hat{p}_1 + \mu e^{\hat{d}} S_{-\mu}(\hat{\theta}) \otimes \hat{p}_2 \\
&\Delta(\hat{p}_2) = \hat{p}_2 \otimes 1 + e^{\hat{d}} C_{-\mu}(\hat{\theta}) \otimes \hat{p}_2 + e^{\hat{d}} S_{-\mu}(\hat{\theta}) \otimes \hat{p}_1 \\
&\Delta(\hat{c}_1) = \hat{c}_1 \otimes 1 + e^{-\hat{d}} C_{-\mu}(\hat{\theta}) \otimes \hat{c}_1 + \mu e^{-\hat{d}} S_{-\mu}(\hat{\theta}) \otimes \hat{c}_2 \\
&\Delta(\hat{c}_2) = \hat{c}_2 \otimes 1 + e^{-\hat{d}} C_{-\mu}(\hat{\theta}) \otimes \hat{c}_2 + e^{-\hat{d}} S_{-\mu}(\hat{\theta}) \otimes \hat{c}_1
\end{aligned} \tag{3.17}$$

$$\epsilon(X) = 0 \qquad X \in \{\hat{\theta}, \hat{p}_i, \hat{c}_i, \hat{d}\} \tag{3.18}$$

$$\begin{aligned}
&\gamma(\hat{\theta}) = -\hat{\theta} \qquad \gamma(\hat{d}) = -\hat{d} \\
&\gamma(\hat{p}_1) = -e^{-\hat{d}} C_{-\mu}(\hat{\theta})\hat{p}_1 - \mu e^{-\hat{d}} S_{-\mu}(\hat{\theta})\hat{p}_2 \\
&\gamma(\hat{p}_2) = -e^{-\hat{d}} C_{-\mu}(\hat{\theta})\hat{p}_2 - e^{-\hat{d}} S_{-\mu}(\hat{\theta})\hat{p}_1 \\
&\gamma(\hat{c}_1) = -e^{\hat{d}} C_{-\mu}(\hat{\theta})\hat{c}_1 - \mu e^{\hat{d}} S_{-\mu}(\hat{\theta})\hat{c}_2 \\
&\gamma(\hat{c}_2) = -e^{\hat{d}} C_{-\mu}(\hat{\theta})\hat{c}_2 - e^{\hat{d}} S_{-\mu}(\hat{\theta})\hat{c}_1.
\end{aligned} \tag{3.19}$$

The final step in this quantization process consists of deducing the universal R-matrix for $U_w g_\mu$. We write two R-matrices (2.29) R^1_z and R^2_{-z} with generators $\{K^l, P^l_\pm\}$ $(l = 1, 2)$ for the two copies $U_{\pm z} iso(1,1)$ and compute the product $\mathcal{R} = R^1_z R^2_{-z}$:

$$\begin{aligned}
\mathcal{R} &= \exp\left\{K^1 \wedge \sinh zP^1_+ \frac{z\Delta P^1_+}{\sinh z\Delta P^1_+}\right\} \exp\left\{-K^2 \wedge \sinh zP^2_+ \frac{z\Delta P^2_+}{\sinh z\Delta P^2_+}\right\} \\
&= \exp\left\{(K^1 \wedge \sinh zP^1_+ \Delta P^1_+ \sinh z\Delta P^2_+ - K^2 \wedge \sinh zP^2_+ \Delta P^2_+ \sinh z\Delta P^1_+) \right. \\
&\qquad \left. \times \frac{z}{\sinh z\Delta P^1_+ \sinh z\Delta P^2_+}\right\}.
\end{aligned} \tag{3.20}$$

We introduce the change of generators (3.1) and then apply the quantum contraction (3.4). The final expression for the universal R-matrix of $U_w g_\mu$ (denoted by $\mathcal{R}_w$) is

$$\mathcal{R}_w = \exp\{(M_1 N_1 + M_2 N_2)L\} \tag{3.21}$$

where

$$\begin{aligned}
M_1 &= D \wedge \mathrm{C}_{-\mu}(wP_1/2)\sinh(wP_2/2) + J \wedge \mathrm{S}_{-\mu}(wP_1/2)\cosh(wP_2/2) \\
M_2 &= \mu D \wedge \mathrm{S}_{-\mu}(wP_1/2)\cosh(wP_2/2) + J \wedge \mathrm{C}_{-\mu}(wP_1/2)\sinh(wP_2/2) \\
N_1 &= \mu\Delta P_1 S_{-\mu}(w\Delta P_1/2)\cosh(w\Delta P_2/2) - \Delta P_2 \mathrm{C}_{-\mu}(w\Delta P_1/2)\sinh(w\Delta P_2/2) \\
N_2 &= \Delta P_2 \mathrm{S}_{-\mu}(w\Delta P_1/2)\cosh(w\Delta P_2/2) - \Delta P_1 \mathrm{C}_{-\mu}(w\Delta P_1/2)\sinh(w\Delta P_2/2) \\
L &= \frac{2w}{\mathrm{C}_{-\mu}(w\Delta P_1) - \cosh(w\Delta P_2)}.
\end{aligned} \tag{3.22}$$

An interesting idea naturally arising from this result would be the use of the FRT construction [13] to quantize $Fun(G_\mu)$. In fact, the matrix representation (3.5) substituted in (3.21) gives rise to a particular representation of $\mathcal{R}_w$:

$$\begin{aligned}
\mathcal{R}_w &= \exp\{wr\} = \exp\{w(J \wedge P_1 + D \wedge P_2)\} \\
&= I \otimes I + w(J \wedge P_1 + D \wedge P_2) + \mu w^2 P_1 \otimes P_1
\end{aligned} \tag{3.23}$$

where I is the four-dimensional identity matrix. In this representation the commutation rules of the group coordinates $(\hat{d}, \hat{\theta}, \hat{p}_i, \hat{c}_i)$ would be deduced from the equation

$$\mathcal{R}_w T_1 T_2 = T_2 T_1 \mathcal{R}_w \tag{3.24}$$

where T is the generic element of the group G_μ (3.6), $T_1 = T \otimes I$ and $T_2 = I \otimes T$. Lengthy computations show that commutators obtained in this way are exactly those given in (3.16) up to a global change of sign in the deformation parameter w. Furthermore, coproduct (3.17), co-unit (3.18) and antipode (3.19) can be obtained from the relations $\Delta(T) = T\dot{\otimes}T$, $\epsilon(T) = I$ and $\gamma(T) = T^{-1}$.

4. Universal quantizations of Weyl subalgebras

The Lie brackets

$$[J, P_1] = P_2 \qquad [J, P_2] = \mu P_1 \qquad [P_1, P_2] = 0 \tag{4.1}$$

correspond for μ negative, positive and zero to the two-dimensional Euclidean, Poincaré and Galilei algebras respectively. We can enlarge these algebras by means of a dilation generator D:

$$[D, P_i] = P_i \qquad [D, J] = 0. \tag{4.2}$$

These enlarged algebras are the similitude algebras of the Euclidean, Minkowskian or Galilean planes, and will be denoted by s_μ; they are the Weyl subalgebras of the corresponding conformal algebras in two dimensions. Although the conformal algebras of the family (4.1) are indeed $so(3,1)$, $iso(2,1)$ and $so(2,2)$ for $\mu < 0$, $\mu = 0$ and $\mu > 0$ respectively [1], the crucial point is that each of the algebras in the family g_μ also contains a subalgebra isomorphic to the Weyl subalgebra (4.1) and (4.2). Moreover, the Hopf algebra $U_w g_\mu$ preserves this property, that is, $U_w g_\mu$ includes quantum Weyl subalgebras that deform (4.1) and (4.2). Therefore, the following proposition follows.

Proposition 5. The algebras $U_w s_\mu$ given in [1]:

$$
\begin{aligned}
&\Delta P_1 = 1\otimes P_1 + P_1\otimes 1 \qquad \Delta P_2 = 1\otimes P_2 + P_2\otimes 1\\
&\Delta J = \mathrm{e}^{-\frac{w}{2}P_2}\mathrm{C}_{-\mu}(wP_1/2)\otimes J + J\otimes \mathrm{C}_{-\mu}(wP_1/2)\mathrm{e}^{\frac{w}{2}P_2} - \mathrm{e}^{-\frac{w}{2}P_2}\mathrm{S}_{-\mu}(wP_1/2)\otimes \mu D\\
&\qquad +\mu D\otimes \mathrm{S}_{-\mu}(wP_1/2)\mathrm{e}^{\frac{w}{2}P_2}\\
&\Delta D = \mathrm{e}^{-\frac{w}{2}P_2}\mathrm{C}_{-\mu}(wP_1/2)\otimes D + D\otimes \mathrm{C}_{-\mu}(wP_1/2)\mathrm{e}^{\frac{w}{2}P_2} - \mathrm{e}^{-\frac{w}{2}P_2}\mathrm{S}_{-\mu}(wP_1/2)\otimes J\\
&\qquad +J\otimes \mathrm{S}_{-\mu}(wP_1/2)\mathrm{e}^{\frac{w}{2}P_2}\\
&\epsilon(X)=0 \qquad \gamma(X) = -\mathrm{e}^{wP_2}X\mathrm{e}^{-wP_2} \qquad X\in\{J, P_i, D\} \qquad (4.3)\\
&[J,P_1] = \frac{2}{w}\sinh(wP_2/2)\mathrm{C}_{-\mu}(wP_1/2)\\
&[J,P_2] = \frac{2}{w}\mu \mathrm{S}_{-\mu}(wP_1/2)\cosh(wP_2/2)\\
&[D,P_1] = \frac{2}{w}\mathrm{S}_{-\mu}(wP_1/2)\cosh(wP_2/2)\\
&[D,P_2] = \frac{2}{w}\sinh(wP_2/2)\mathrm{C}_{-\mu}(wP_1/2)\\
&[P_1,P_2]=0 \qquad [D,J]=0
\end{aligned}
$$

are quasi-triangular Hopf algebras with the universal R-matrix (3.21) and (3.22).

Furthermore, it is clear that by taking the generators $C_i \equiv 0$ and the group parameters $\hat{c}_i \equiv 0$ in proposition 3 and 4 we find a Poisson–Hopf algebra structure for $Fun(S_\mu)$ and a quantum Hopf algebra $Fun_w(S_\mu)$.

5. Concluding remarks

A combined approach of the construction $U_zA \oplus U_{-z}A$ (A being either a Lie algebra or the algebra of functions on the Lie group) together with a quantum contraction provide a simultaneous universal quantization for the algebras g_μ in the family (3.3).

One of the groups in the family G_μ can be realized as a kinematical group: the group $G_{-1} \equiv T_4(SO(2)\otimes SO(1,1))$ is isomorphic to the $(2+1)$ expanding Newton–Hooke group [14], the motion group of a non-relativistic space-time with constant negative curvature. Time is absolute in such a universe and a space-time contraction leads to the Galilean case. An adapted basis for G_{-1} is formed by a time translation $\tilde{H}$, two spatial translations $\tilde{P}_i$, two boosts $\tilde{K}_i$ and one spatial rotation $\tilde{J}$, with corresponding group coordinates $\{t, x_i, v_i, \psi\}$ $(i=1,2)$. All expressions obtained for G_{-1} in section 3 can be written in terms of these new generators and group coordinates by means of the isomorphisms

$$\tilde{J}\equiv J \qquad \tilde{P}_i \equiv \tfrac{1}{2}(P_i + C_i) \qquad \tilde{K}_i \equiv \tfrac{1}{2}(P_i - C_i) \qquad \tilde{H}\equiv -D \qquad (5.1)$$

$$\psi\equiv\theta \qquad x_i \equiv 2(p_i + c_i) \qquad v_i \equiv 2(p_i - c_i) \qquad t\equiv -d. \qquad (5.2)$$

An open problem still to be solved is the construction of a universal R-matrix for the non-standard quantum deformation of $sl(2,\mathbb{R})$ which would provide a set of universal R-matrices for the whole set of $(2+1)$ non-standard quantum algebras, following the method just described. We recall that among this set there are some rather interesting cases from a physical point of view: the conformal algebras of the $(1+1)$ Poincaré ($so(2,2)$) and two-dimensional Euclidean spaces ($so(3,1)$), besides a 'null-plane' $(2+1)$ Poincaré algebra.

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